

On the reduction in remaining system time above a a specific state due to a jump downwards of a component in nonrepairable multistate strongly coherent systems

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Abstract In the present paper results given in Natvig (1990) are generalized to a multistate, strongly coherent, nonrepairable system of independent components by considering the reduction in remaining system time above a certain state due to a jump downwards of a component. This reduction also equals the increase in remaining system time above a certain state due to a minimal repair of the component at its time of jump downwards. The expected value of the sum of such reductions/increases for the different possible jumps downwards of the component is the building block of the Natvig measure of the importance of the component in the multistate case. Hence, now the whole distributions of these reductions/increases are arrived at, not only the expectations, then throwing more light on the consequences for the system of the deterioration of the components.

Keywords: Dynamic reliability; Natvig importance of a system component; multistate strongly coherent systems; nonrepairable systems

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1. Introduction

In Natvig (1990) the reduction in remaining system lifetime due to the failure of a specific component and a specific module in a binary, coherent, nonrepairable system of independent components was considered. The former reduction also equals the increase in remaining system lifetime due to a minimal repair of the component at its time of failure. The expected value of this

reduction/increase is the building block of the so-called Natvig measure of the importance of the component as treated in Natvig (1979), (1982a), (1985) and Natvig and Gåsemyr (2009). Hence, in Natvig (1990) the whole distribution of this reduction/increase was arrived at not only the expectation. In the present paper these results are generalized to a multistate, strongly coherent, nonrepairable system of independent components by considering the reduction in remaining system time above a certain state due to a jump downwards of a component. Again this reduction also equals the increase in remaining system time above a certain state due to a minimal repair of the component at its time of jump downwards. The expected value of the sum of such reductions/increases for the different possible jumps downwards of the component is the building block of the Natvig measure of the importance of the component in the multistate case as treated in Natvig (2011a). Hence, now the whole distributions of these reductions/increases are arrived at, not only the expectations, then throwing more light on the consequences for the system of the deterioration of the components.

Let $S = \{0, 1, \dots, M\}$ be the set of states of the system; the $M + 1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0 . Furthermore, let $C = \{1, \dots, n\}$ be the set of components and in general $S_i, i = 1, \dots, n$ the set of states of the i th component. We claim $\{0, M\} \subseteq S_i \subseteq S$. Hence, the states 0 and M are chosen to represent the endpoints of a performance scale that might be used for both the system and its components. Note that in most applications there is no need for the same detailed description of the components as for the system.

Let $x_i, i = 1, \dots, n$ denote the state or performance level of the i th component at a fixed point of time and $\mathbf{x} = (x_1, \dots, x_n)$. It is assumed that the state, ϕ , of the system at the fixed point of time is a deterministic function of \mathbf{x} ; i.e. $\phi = \phi(\mathbf{x})$. Here \mathbf{x} takes values in $S_1 \times S_2 \times \dots \times S_n$ and ϕ takes values in S . The function ϕ is called the structure function of the system. We often denote a multistate system by (C, ϕ) .

Definition 1 *A system is a multistate monotone system (MMS) iff its structure function ϕ satisfies:*

- (i) $\phi(\mathbf{x})$ is non-decreasing in each argument
- (ii) $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{M}) = M$ $\mathbf{0} = (0, \dots, 0), \mathbf{M} = (M, \dots, M)$

Let

$$(\cdot, \mathbf{x}) = (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$$

Now choose $j \in \{1, \dots, M\}$ and let the states $\{0, \dots, j - 1\}$ correspond to the failure state and $\{j, \dots, M\}$ to the functioning state if a binary approach had been applied. Following this approach it seems natural, for any way of distinguishing between the binary failure and functioning state, to claim each

component to be relevant. More precisely for any $j \in \{1, \dots, M\}$ and any component i , there should exist a vector (\cdot, \mathbf{x}) such that if the i th component is in the binary failure state, the system itself is in the binary failure state and correspondingly if the i th component is in the binary functioning state, the system itself is in the binary functioning state. This motivates the following definition of a multistate strongly coherent system, which for the case $S_i = S$, $i = 1, \dots, n$ is introduced as a multistate coherent system of type 1 in Natvig (1982b).

The following notation is needed

$$S_{i,j}^0 = S_i \cap \{0, \dots, j-1\} \quad \text{and} \quad S_{i,j}^1 = S_i \cap \{j, \dots, M\} \quad (1)$$

Definition 2 Consider an MMS with structure function ϕ satisfying

$$(i) \quad \min_{1 \leq i \leq n} x_i \leq \phi(\mathbf{x}) \leq \max_{1 \leq i \leq n} x_i,$$

where $\min_{1 \leq i \leq n} x_i$ and $\max_{1 \leq i \leq n} x_i$ are respectively the multistate series and parallel structure functions. If in addition $\forall i \in \{1, \dots, n\}$, $\forall j \in \{1, \dots, M\}$, $\exists (\cdot, \mathbf{x})$ such that

$$(ii) \quad \phi(k_i, \mathbf{x}) \geq j, \quad \phi(\ell_i, \mathbf{x}) < j, \quad \forall k \in S_{i,j}^1, \quad \forall \ell \in S_{i,j}^0, \quad \text{we have a multistate strongly coherent system (MSCS).}$$

In this paper we will concentrate on multistate strongly coherent systems. We now consider the relation between the stochastic performance of the system (C, ϕ) and the stochastic performances of the components. Introduce the random state $X_i(t)$ of the i th component at time t , $i = 1, \dots, n$ and the corresponding random vector $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$. Now if ϕ is a multistate structure function, $\phi(\mathbf{X}(t))$ is the corresponding random system state at time t . Assume also that the stochastic processes $\{X_i(t), t \in [0, \infty)\}$, $i = 1, \dots, n$, are mutually independent. For the dynamic approach of the present paper this is a necessary assumption in order to arrive at explicit results.

We restrict our attention to the case where the components, and hence the system, cannot be repaired. In order to avoid a rather complex notation we will in the following assume that $S_i = S$, $i = 1, \dots, n$. Furthermore, assume that at time $t = 0$ all components are in the perfect functioning state M ; i.e. $\mathbf{X}(0) = \mathbf{M}$. Introduce the notation

$$\begin{aligned} P(X_i(t) = j) &= r_i^j(t), \quad j = 0, \dots, M \\ \mathbf{r}(t) &= (r_1^1(t), \dots, r_1^M(t), r_2^1(t), \dots, r_n^M(t)) \\ p_i^{k,\ell}(t, t+u) &= P(X_i(t+u) = \ell \mid X_i(t) = k), \quad 0 \leq \ell < k \leq M \\ \lambda_i^{k,\ell}(t) &= \lim_{h \rightarrow 0} p_i^{(k,\ell)}(t, t+h)/h, \quad 0 \leq \ell < k \leq M \end{aligned}$$

$$P[\phi(\mathbf{X}(t)) \geq j] = P[I(\phi(\mathbf{X}(t)) \geq j) = 1] = p_\phi^j(\mathbf{r}(t)),$$

where $I(\cdot)$ is the indicator function. $p_\phi^j(\mathbf{r}(t))$ is the reliability to level j of the system at time t .

In order to make things not too complex we assume that

$$\lambda_i^{k,\ell}(t) = 0, \quad 0 \leq \ell < k - 1 \leq M - 1$$

Hence, each component deteriorates by going through all states from the perfect functioning state until the complete failure state. Let the i th component have an absolutely continuous distribution $F_i^k(t)$ of time spent in state k , before jumping downwards to state $k - 1$, with density $f_i^k(t)$ and $\bar{F}_i^k(t) = 1 - F_i^k(t)$. It is assumed that all these times spent in the various states are independent. Hence, $X_i(t), i = 1, \dots, n$ for $t \in [0, \infty)$, are semi-Markov processes in continuous time, not Markov processes as stated in Natvig (2011a,b).

The paper is organized as follows. In Section 2 the reduction in remaining system time above state $j \in \{1, \dots, M\}$ due to a jump downwards of the i th component from state $k \in \{1, \dots, M\}$ is considered. Corresponding results for a module are given in Section 3. Finally, some concluding remarks are given in Section 4.

2. Reduction in Remaining System Time above a Specific State due to a Jump Downwards of a Component

Intuitively it seems that components that by deteriorating, strongly reduce the expected remaining system time in the better states, are very important. This seems at least true during the system development phase. This is the motivation for the following generalization to multistate systems, given in Natvig (2011a), of the Natvig (1979) measure of the importance of the i th component. In Natvig (2011a), as in the binary case treated in Natvig (1982a), the actual definition of this generalized measure is in terms of the effect on future system performance of a fictive minimal repair of a component.

We introduce for $i = 1, \dots, n, k \in \{0, \dots, M - 1\}$

$T_{i,k}$ = the time of the jump of the i th component into state k .

$T'_{i,k}$ = the fictive time of the jump of the i th component into state k

after a fictive minimal repair of the component at $T_{i,k}$; i.e. it is repaired to have the same distribution of remaining time in state $k + 1$ as it had just

before jumping downwards to state k .

Furthermore, for $i = 1, \dots, n, k \in \{1, \dots, M\}, j \in \{1, \dots, M\}$ we introduce

$$Z_{i,k,j} = Y_{i,k,j}^1 - Y_{i,k,j}^0, \quad (2)$$

where

$Y_{i,k,j}^1$ = system time in the states $\{j, \dots, M\}$ in the interval $[T_{i,k-1}, T'_{i,k-1}]$ just *after* the jump downwards from state k to state $k - 1$ of the i th component, which, however, immediately undergoes a fictive minimal repair.

$Y_{i,k,j}^0$ = system time in the states $\{j, \dots, M\}$ in the interval $[T_{i,k-1}, T'_{i,k-1}]$ just *after* the jump downwards from state k to state $k - 1$ of the i th component, assuming that the component stays in the latter state throughout this interval.

Thus, $Z_{i,k,j}$ is the fictive increase in system time in the states $\{j, \dots, M\}$ in the interval $[T_{i,k-1}, T'_{i,k-1}]$ due to a fictive minimal repair of the i th component when jumping downwards from state k to state $k - 1$. Note that since the minimal repair is fictive, we have chosen to calculate the effect of this repair over the entire interval $[T_{i,k-1}, T'_{i,k-1}]$ even though this interval may extend beyond the time of the next jump of the i th component. Note that the fictive minimal repair periods; i.e. the intervals of the form $[T_{i,k-1}, T'_{i,k-1}]$, may sometimes overlap. Thus, at a given point of time we may have contributions from more than one fictive minimal repair. This was efficiently dealt with by the simulation methods presented in Huseby and Natvig (2012). Taking the expectation, we get for $i = 1, \dots, n, j \in \{1, \dots, M\}$ the following generalized Natvig measure, $I_N^{(i,j)}$, of the importance of the i th component

$$I_N^{(i,j)} = \sum_{k=1}^M EZ_{i,k,j} / \sum_{r=1}^n \sum_{k=1}^M EZ_{r,k,j}, \quad (3)$$

tacitly assuming $EZ_{i,k,j} < \infty$, $i = 1, \dots, n, k \in \{1, \dots, M\}, j \in \{1, \dots, M\}$. We obviously have

$$\sum_{i=1}^n I_N^{(i,j)} = 1, \quad 0 \leq I_N^{(i,j)} \leq 1 \quad (4)$$

Lemma 1 and 2 below are given without proofs except that noting that the terms $(\bar{F}_i^k(z+u)/\bar{F}_i^k(z))$ and $(\bar{F}_i^k(z+v)/\bar{F}_i^k(v))$ in Lemma 1 are entering since we are considering the interval $[T_{i,k-1}, T'_{i,k-1}]$. These lemmas are generalizations of Lemma 2.1 in Natvig (1982a) covering a binary coherent system.

Lemma 1 For $k \in \{1, \dots, M-1\}$

$$\begin{aligned}
\bar{G}_{i,k,j}^1(u) &= P[Y_{i,k,j}^1 > u] = \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_k)_i, \mathbf{r}(s_i + z + u)\right) \\
&\quad r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) (\bar{F}_i^k(z+u) / \bar{F}_i^k(z)) dz ds_i \\
\bar{G}_{i,M,j}^1(u) &= P[Y_{i,M,j}^1 > u] = \int_0^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_M)_i, \mathbf{r}(z+u)\right) \\
&\quad f_i^M(z) (\bar{F}_i^M(z+u) / \bar{F}_i^M(z)) dz \\
\bar{G}_{i,k,j}^0(v) &= P[Y_{i,k,j}^0 > v] = \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_{k-1})_i, \mathbf{r}(s_i + z + v)\right) \\
&\quad r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) (\bar{F}_i^k(z+v) / \bar{F}_i^k(z)) dz ds_i \\
\bar{G}_{i,M,j}^0(v) &= P[Y_{i,M,j}^0 > v] = \int_0^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_{M-1})_i, \mathbf{r}(z+v)\right) \\
&\quad f_i^M(z) (\bar{F}_i^M(z+v) / \bar{F}_i^M(z)) dz
\end{aligned}$$

From Lemma 1 we arrive at the following expression for the probability that the jump downwards of the i th component from state $k \in \{1, \dots, M-1\}$ to state $k-1$ leads to the system leaving the states $\{j, \dots, M\}$.

$$\begin{aligned}
P[Y_{i,k,j}^1 > 0, Y_{i,k,j}^0 = 0] &= P[Y_{i,k,j}^0 = 0] - P[Y_{i,k,j}^1 = 0, Y_{i,k,j}^0 = 0] \\
&= P[Y_{i,k,j}^0 = 0] - P[Y_{i,k,j}^1 = 0] = \bar{G}_{i,k,j}^1(0) - \bar{G}_{i,k,j}^0(0) \\
&= \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_k)_i, \mathbf{r}(s_i + z)\right) r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) dz ds_i \\
&\quad - \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_{k-1})_i, \mathbf{r}(s_i + z)\right) r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) dz ds_i \\
&= \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} I_B^{(i,k,j)}(s_i + z) r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) dz ds_i \\
&= \int_0^{\infty} I_B^{(i,k,j)}(t) r_i^k(t) \lambda_i^{k,k-1}(t) dt, \tag{5}
\end{aligned}$$

where

$$I_B^{(i,k,j)}(t) = p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_k)_i, \mathbf{r}(t)\right) - p_{\phi}^j\left(\mathbf{0}, \mathbf{1}_{k-1})_i, \mathbf{r}(t)\right)$$

The same expressions follow easily for $k = M$. The latter expression is the generalized Birnbaum (1969) measure of the importance of the i th component at time t given in Natvig (2011a) being the probability that the

system is in the states $\{j, \dots, M\}$ if at time t the i th component is in state k and not if the i th component is in state $k - 1$. By summing (5) from $k = 1$ to $k = M$ one arrives at the generalized Barlow-Proschan (1975) measure of the importance of the i th component given in Natvig (2011a) being the probability that the jump downwards of the i th component coincides with the system leaving the states $\{j, \dots, M\}$.

Note also that the probability that the system is leaving the states $\{j, \dots, M\}$ before the jump downwards of the i th component from state $k \in \{1, \dots, M\}$ to state $k - 1$ is given by

$$P[Y_{i,k,j}^1 = 0] = 1 - \bar{G}_{i,k,j}^1(0) = 1 - \int_0^\infty p_\phi^j\left((\mathbf{0}, 1_k)_i, \mathbf{r}(t)\right) r_i^k(t) \lambda_i^{k,k-1}(t) dt$$

Lemma 2 For $k \in \{1, \dots, M - 1\}$, $0 \leq v \leq u$

$$\begin{aligned} \bar{G}_{i,k,j}(u, v) &= P[Y_{i,k,j}^1 > u, Y_{i,k,j}^0 > v] \\ &= \int_{s_i=0}^\infty \int_{z=0}^\infty \sum_{(\cdot, \mathbf{x})} \sum_{(\cdot, \mathbf{y}) \leq (\cdot, \mathbf{x})} I(\phi((k-1)_i, \mathbf{x}) \geq j) I(\phi(k_i, \mathbf{y}) \geq j) \\ &\quad \prod_{l \neq i} [r_l^{x_l}(s_i + z + v) p_l^{x_l, y_l}(s_i + z + v, s_i + z + u)] \\ &\quad r_i^{k+1}(s_i) \lambda_i^{k+1, k}(s_i) f_i^k(z) (\bar{F}_i^k(z+u) / \bar{F}_i^k(z)) dz ds_i \\ \bar{G}_{i,M,j}(u, v) &= P[Y_{i,M,j}^1 > u, Y_{i,M,j}^0 > v] \\ &= \int_{z=0}^\infty \sum_{(\cdot, \mathbf{x})} \sum_{(\cdot, \mathbf{y}) \leq (\cdot, \mathbf{x})} I(\phi((M-1)_i, \mathbf{x}) \geq j) I(\phi(M_i, \mathbf{y}) \geq j) \\ &\quad \prod_{l \neq i} [r_l^{x_l}(z + v) p_l^{x_l, y_l}(z + v, z + u)] f_i^M(z) (\bar{F}_i^M(z+u) / \bar{F}_i^M(z)) dz \end{aligned}$$

The distribution of $Z_{i,k,j}$ is given by the following theorem. The proof for $P(Z_{i,k,j} = 0)$ follows partly from (5). The proof for $P(Y_{i,k,j}^1 > Y_{i,k,j}^0 > 0)$ is not based on minimal cut sets containing the i th component, as in Natvig (1990) treating the binary case, since these sets may be identical in the multistate case even if the minimal cut vectors are different. The proof for the absolutely continuous part is completely parallel to the one given in Theorem 2.3 of Natvig (1982a) now inserting the expressions for $\bar{G}_{i,k,j}^1(u)$ and $\bar{G}_{i,k,j}(u, v)$ from Lemma 1 and 2.

Theorem 1. For $k \in \{1, \dots, M - 1\}$

$$P(Z_{i,k,j} = 0) = 1 - P(Z_{i,k,j} > 0) = 1 - P(Y_{i,k,j}^1 > Y_{i,k,j}^0 > 0) - P(Y_{i,k,j}^1 > 0,$$

$$\begin{aligned}
Y_{i,k,j}^0 = 0) &= 1 - \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} \int_{v=0}^{\infty} r_i^{k+1}(s_i) \lambda_i^{k+1,k}(s_i) f_i^k(z) (\bar{F}_i^k(z+v)/\bar{F}_i^k(z)) \\
&\sum_{(\cdot, \mathbf{x})} \prod_{h \neq i} r_h^{x_h}(s_i+z+v) \sum_{l \neq i} I(x_l > 0) \lambda_l^{x_l, x_l-1}(s_i+z+v) \\
&[I(\phi((k-1)_i, (x_l)_l, \mathbf{x}) \geq j) I(\phi(k_i, (x_l-1)_l, \mathbf{x}) \geq j) \\
&- I(\phi((k-1)_i, (x_l-1)_l, \mathbf{x}) \geq j)] dv dz ds_i - \int_0^{\infty} I_B^{(i,k,j)}(t) r_i^k(t) \lambda_i^{k,k-1}(t) dt \\
P(Z_{i,M,j} = 0) &= 1 - P(Z_{i,M,j} > 0) = 1 - P(Y_{i,M,j}^1 > Y_{i,M,j}^0 > 0) - P(Y_{i,M,j}^1 > 0, \\
Y_{i,M,j}^0 = 0) &= 1 - \int_{z=0}^{\infty} \int_{v=0}^{\infty} f_i^M(z) (\bar{F}_i^M(z+v)/\bar{F}_i^M(z)) \\
&\sum_{(\cdot, \mathbf{x})} \prod_{h \neq i} r_h^{x_h}(z+v) \sum_{l \neq i} I(x_l > 0) \lambda_l^{x_l, x_l-1}(z+v) \\
&[I(\phi((M-1)_i, (x_l)_l, \mathbf{x}) \geq j) I(\phi(M_i, (x_l-1)_l, \mathbf{x}) \geq j) \\
&- I(\phi((M-1)_i, (x_l-1)_l, \mathbf{x}) \geq j)] dv dz - \int_0^{\infty} I_B^{(i,M,j)}(t) r_i^M(t) \lambda_i^{M,M-1}(t) dt
\end{aligned}$$

Furthermore, let

$$\begin{aligned}
g_{i,k,j}(u, v) &= \partial^2 \bar{G}_{i,k,j}(u, v) / \partial u \partial v, \quad 0 < v < u \\
g_{i,k,j}^1(u, 0) &= \partial [\bar{G}_{i,k,j}(u, 0) - \bar{G}_{i,k,j}^1(u)] / \partial u, \quad 0 < u
\end{aligned}$$

Then the absolutely continuous part of the distribution of $Z_{i,k,j}$ has density

$$g_{i,k,j}(z) = g_{i,k,j}^1(z, 0) + \int_0^{\infty} g_{i,k,j}(v+z, v) dv, \quad 0 < z$$

To illustrate the theory we now consider the multistate series and parallel systems as given in Natvig (2011b) generalizing results from Natvig (1982a) covering the binary case. For the multistate series system $\phi(\mathbf{x}) = \min_{1 \leq i \leq n} x_i$. We consider the case $k = j$ for $j \in \{1, \dots, M-1\}$. Now obviously $Z_{i,j,j} = Y_{i,j,j}^1$. From Lemma 1 and 2 we get

$$\begin{aligned}
\bar{G}_{i,j,j}^1(u) &= \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} \prod_{l \neq i} \sum_{m=j}^M r_l^m(s_i+z+u) r_i^{j+1}(s_i) \lambda_i^{j+1,j}(s_i) \\
&f_i^j(z) (\bar{F}_i^j(z+u)/\bar{F}_i^j(z)) dz ds_i \\
\bar{G}_{i,j,j}^0(v) &= 0
\end{aligned}$$

$$\bar{G}_{i,j,j}(u, v) = 0$$

From Theorem 1 we get

$$P(Z_{i,j,j} = 0) = 1 - \int_0^\infty \prod_{l \neq i} \sum_{m=j}^M r_l^m(t) r_i^j(t) \lambda_i^{j,j-1}(t) dt$$

We now turn to the multistate parallel system where $\phi(\mathbf{x}) = \max_{1 \leq i \leq n} x_i$. Again we consider the case $k = j$ for $j \in \{1, \dots, M-1\}$. Introduce the notation $\prod_{l \in A} x_l = 1 - \prod_{l \in A} (1 - x_l)$. From Lemma 1 and 2 we get

$$\begin{aligned} \bar{G}_{i,j,j}^1(u) &= \int_{s_i=0}^\infty \int_{z=0}^\infty r_i^{j+1}(s_i) \lambda_i^{j+1,j}(s_i) f_i^j(z) (\bar{F}_i^j(z+u)/\bar{F}_i^j(z)) dz ds_i \\ \bar{G}_{i,j,j}^0(v) &= \int_{s_i=0}^\infty \int_{z=0}^\infty \prod_{l \neq i} \sum_{m=j}^M r_l^m(s_i+z+v) \\ &\quad r_i^{j+1}(s_i) \lambda_i^{j+1,j}(s_i) f_i^j(z) (\bar{F}_i^j(z+v)/\bar{F}_i^j(z)) dz ds_i \\ \bar{G}_{i,j,j}(u, v) &= \int_{s_i=0}^\infty \int_{z=0}^\infty \sum_{(\cdot, \mathbf{x})} \prod_{h \neq i} I(x_h \geq j) \prod_{l \neq i} r_l^{x_l}(s_i+z+v) \\ &\quad r_i^{j+1}(s_i) \lambda_i^{j+1,j}(s_i) f_i^j(z) (\bar{F}_i^j(z+u)/\bar{F}_i^j(z)) dz ds_i \end{aligned}$$

From Theorem 1 we get

$$\begin{aligned} P(Z_{i,j,j} = 0) &= 1 - \int_{s_i=0}^\infty \int_{z=0}^\infty \int_{v=0}^\infty r_i^{j+1}(s_i) \lambda_i^{j+1,j}(s_i) f_i^j(z) (\bar{F}_i^j(z+v)/\bar{F}_i^j(z)) \\ &\quad \sum_{l \neq i} \prod_{h \neq i, l} \sum_{m=0}^{j-1} r_h^m(s_i+z+v) r_l^j(s_i+z+v) \lambda_l^{j,j-1}(s_i+z+v) \} dv dz ds_i \\ &\quad - \int_0^\infty \prod_{h \neq i} \sum_{m=0}^{j-1} r_h^m(t) r_i^j(t) \lambda_i^{j,j-1}(t) dt \end{aligned}$$

3. Reduction in Remaining System Time above a Specific State due to a Jump Downwards of a Component inside a Module

Let the multistate strongly coherent system have the modular decomposition $\{M_g, \chi_g\}_{g=1}^a$ being defined in the same way both in binary theory by Barlow

and Proschan (1981) and in multistate theory by Natvig (2011b). Introduce the random variable $Z_{M_g,b,j}$ being the fictive increase in remaining system time above state j due to a fictive minimal repair of the g th module at its time of jump downwards from states $\{b, \dots, M\}$ to $\{0, \dots, b-1\}$ where $b \in \{1, \dots, M\}$. Since a module consists of more than one component, we feel that this minimal repair should not be of the "black box" type. Having in mind what is going on physically, the minimal repair of the module should rather be interpreted as a "black box" minimal repair of the component in the module that "caused" the module making such a jump downwards by itself jumping downwards. This was done in Natvig (1979), (1982a), (1990) and will also be the approach in the present paper. Especially, in the multistate case we get a contribution to the distribution of $Z_{M_g,b,j}$ for all components $i \in M_g$ and all jumps downwards for these components from state k to state $k-1$ for $k \in \{1, \dots, M\}$.

Let

$$Z_{M_g,b,j} = Y_{M_g,b,j}^1 - Y_{M_g,b,j}^0, \quad (6)$$

where

$Y_{M_g,b,j}^1$ = system time in the states $\{j, \dots, M\}$ in the say interval $[T_{i,k-1}, T'_{i,k-1}]$ just *after* the jump downwards from say state k to state $k-1$ of say the i th component being inside M_g , and also of M_g jumping downwards from $\{b, \dots, M\}$ to $\{0, \dots, b-1\}$. The component and hence also M_g , however, immediately undergoes a fictive minimal repair.

$Y_{M_g,b,j}^0$ = system time in the states $\{j, \dots, M\}$ in the say interval $[T_{i,k-1}, T'_{i,k-1}]$ just *after* the jump downwards from say state k to state $k-1$ of say the i th component being inside M_g , and also of M_g jumping downwards from $\{b, \dots, M\}$ to $\{0, \dots, b-1\}$, assuming that the component stays in the latter state throughout this interval.

Let I be the random component being inside M_g making a jump downwards from the random state K to $K-1$ when M_g is jumping downwards from $\{b, \dots, M\}$ to $\{0, \dots, b-1\}$. Hence, $Z_{M_g,b,j} = Z_{I,K,j}$.

Lemma 3 and 4 below, given without proofs, are generalizations of a part of Theorem 2.6 in Natvig (1982a) covering a binary coherent system.

Lemma 3

$$\bar{G}_{M_g,b,j}^1(u) = P[Y_{M_g,b,j}^1 > u] = \sum_{i \in M_g} \left\{ \sum_{k \in \{1, \dots, M-1\}} \int_{s_i=0}^{\infty} \int_{z=0}^{\infty}$$

$$\begin{aligned}
& \sum_{(\cdot, (M_g)^c \cup \{i\}, \mathbf{w})} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(k_i, \mathbf{w}) \geq b, \chi_g((k-1)_i, \mathbf{w}) < b] \\
& \prod_{h \in M_g - \{i\}} [r_h^{w_h}(s_i + z) p_h^{w_h, x_h}(s_i + z, s_i + z + u)] r_i^{k+1}(s_i) \lambda_i^{k+1, k}(s_i) f_i^k(z) \\
& (\bar{F}_i^k(z+u) / \bar{F}_i^k(z)) \sum_{(\cdot, M_g, \mathbf{x})} I(\phi(k_i, \mathbf{x}) \geq j) \prod_{h \in M_g^c} r_h^{x_h}(s_i + z + u) dz ds_i \\
& + \int_{z=0}^{\infty} \sum_{(\cdot, (M_g)^c \cup \{i\}, \mathbf{w})} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(M_i, \mathbf{w}) \geq b, \chi_g((M-1)_i, \mathbf{w}) < b] \\
& \prod_{h \in M_g - \{i\}} r_h^{w_h}(z) p_h^{w_h, x_h}(z, z + u) f_i^M(z) \\
& (\bar{F}_i^M(z+u) / \bar{F}_i^M(z)) \sum_{(\cdot, M_g, \mathbf{x})} I(\phi(M_i, \mathbf{x}) \geq j) \prod_{h \in M_g^c} r_h^{x_h}(z + u) dz \}
\end{aligned}$$

$$\begin{aligned}
\bar{G}_{M_g, b, j}^0(v) &= P[Y_{M_g, b, j}^0 > v] = \sum_{i \in M_g} \left\{ \sum_{k \in \{1, \dots, M-1\}} \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} \right. \\
& \sum_{(\cdot, (M_g)^c \cup \{i\}, \mathbf{w})} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(k_i, \mathbf{w}) \geq b, \chi_g((k-1)_i, \mathbf{w}) < b] \\
& \prod_{h \in M_g - \{i\}} [r_h^{w_h}(s_i + z) p_h^{w_h, x_h}(s_i + z, s_i + z + v)] r_i^{k+1}(s_i) \lambda_i^{k+1, k}(s_i) f_i^k(z) \\
& (\bar{F}_i^k(z+v) / \bar{F}_i^k(z)) \sum_{(\cdot, M_g, \mathbf{x})} I(\phi(k-1)_i, \mathbf{x}) \geq j) \prod_{h \in M_g^c} r_h^{x_h}(s_i + z + v) dz ds_i \\
& + \int_{z=0}^{\infty} \sum_{(\cdot, (M_g)^c \cup \{i\}, \mathbf{w})} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(M_i, \mathbf{w}) \geq b, \chi_g((M-1)_i, \mathbf{w}) < b] \\
& \prod_{h \in M_g - \{i\}} r_h^{w_h}(z) p_h^{w_h, x_h}(z, z + v) f_i^M(z) \\
& (\bar{F}_i^M(z+v) / \bar{F}_i^M(z)) \sum_{(\cdot, M_g, \mathbf{x})} I(\phi(M-1)_i, \mathbf{x}) \geq j) \prod_{h \in M_g^c} r_h^{x_h}(z + v) dz \}
\end{aligned}$$

Furthermore, let for $h \neq i$ and $0 \leq t, 0 \leq v \leq u$

$$p_{h,t,t+v}^{w_h, x_h, y_h}(t+u) = P[X_h(t+u) = y_h | X_h(t) = w_h, X_h(t+v) = x_h]$$

Lemma 4 For $0 \leq v \leq u$

$$\begin{aligned}
\bar{G}_{M_g, b, j}(u, v) &= P[Y_{M_g, b, j}^1 > u, Y_{M_g, b, j}^0 > v] = \sum_{i \in M_g} \left\{ \sum_{k \in \{1, \dots, M-1\}} \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} \right. \\
&\quad \sum_{(\cdot, \mathbf{y}^{M_g}) \leq (\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} \sum_{(\cdot, \mathbf{y}^{M_g}) \leq (\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(k_i, \mathbf{w}) \geq b, \chi_g((k-1)_i, \mathbf{w}) < b] \\
&\quad \prod_{h \in M_g - \{i\}} [r_h^{w_h}(s_i + z) p_h^{w_h, x_h}(s_i + z, s_i + z + v) p_{h, s_i+z, s_i+z+v}^{w_h, x_h, y_h}(s_i + z + u)] \\
&\quad \sum_{(\cdot, M_g, \mathbf{x})} \sum_{(\cdot, M_g, \mathbf{y}) \leq (\cdot, M_g, \mathbf{x})} I(\phi((k-1)_i, \mathbf{x}) \geq j) I(\phi(k_i, \mathbf{y}) \geq j) \\
&\quad \prod_{h \in M_g^c} [r_h^{x_h}(s_i + z + v) p_h^{x_h, y_h}(s_i + z + v, s_i + z + u)] \\
&\quad r_i^{k+1}(s_i) \lambda_i^{k+1, k}(s_i) f_i^k(z) (\bar{F}_i^k(z+u) / \bar{F}_i^k(z)) dz ds_i \\
&\quad + \int_{z=0}^{\infty} \sum_{(\cdot, M_g) \leq (\cdot, \mathbf{w})} \sum_{(\cdot, \mathbf{y}^{M_g}) \leq (\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(M_i, \mathbf{w}) \geq b, \chi_g((M-1)_i, \mathbf{w}) < b] \\
&\quad \prod_{h \in M_g - \{i\}} [r_h^{w_h}(z) p_h^{w_h, x_h}(z, z + v) p_{h, z, z+v}^{w_h, x_h, y_h}(z + u)] \\
&\quad \sum_{(\cdot, M_g, \mathbf{x})} \sum_{(\cdot, M_g, \mathbf{y}) \leq (\cdot, M_g, \mathbf{x})} I(\phi((M-1)_i, \mathbf{x}) \geq j) I(\phi(M_i, \mathbf{y}) \geq j) \\
&\quad \left. \prod_{h \in M_g^c} [r_h^{x_h}(z + v) p_h^{x_h, y_h}(z + v, z + u)] f_i^M(z) (\bar{F}_i^M(z+u) / \bar{F}_i^M(z)) dz \right\}
\end{aligned}$$

An expression for $P(Z_{M_g, b, j} = 0)$ is given by the following theorem, again as in Theorem 1 not based on minimal cut sets containing the i th component. The absolutely continuous part of the distribution of $Z_{M_g, b, j}$ is completely parallel to the one given in Theorem 1 now inserting the expressions for $\bar{G}_{M_g, b, j}^1(u)$ and $\bar{G}_{M_g, b, j}(u, v)$ from Lemma 3 and 4.

Theorem 2.

$$\begin{aligned}
P(Z_{M_g, b, j} = 0) &= 1 - P(Z_{M_g, b, j} > 0) = 1 - P(Y_{M_g, b, j}^1 > Y_{M_g, b, j}^0 > 0) \\
&\quad - P(Y_{M_g, b, j}^1 > 0, Y_{M_g, b, j}^0 = 0) = 1 - \sum_{i \in M_g} \left\{ \sum_{k \in \{1, \dots, M-1\}} \int_{s_i=0}^{\infty} \int_{z=0}^{\infty} \int_{v=0}^{\infty} \right. \\
&\quad \sum_{(\cdot, M_g) \leq (\cdot, \mathbf{w})} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(k_i, \mathbf{w}) \geq b, \chi_g((k-1)_i, \mathbf{w}) < b] \\
&\quad \prod_{h \in M_g - \{i\}} [r_h^{w_h}(s_i + z) p_h^{w_h, x_h}(s_i + z, s_i + z + v)] r_i^{k+1}(s_i) \lambda_i^{k+1, k}(s_i) f_i^k(z) \\
&\quad (\bar{F}_i^k(z+v) / \bar{F}_i^k(z)) \sum_{(\cdot, M_g, \mathbf{x})} \prod_{h \in M_g^c} r_h^{x_h}(s_i + z + v) \sum_{l \neq i} I(x_l > 0) \lambda_l^{x_l, x_l-1}(s_i + z + v)
\end{aligned}$$

$$\begin{aligned}
& [I(\phi((k-1)_i, (x_l)_l, \mathbf{x}) \geq j)I(\phi(k_i, (x_l-1)_l, \mathbf{x}) \geq j) \\
& - I(\phi((k-1)_i, (x_l-1)_l, \mathbf{x}) \geq j)]dvdzds_i \\
& + \int_{z=0}^{\infty} \int_{v=0}^{\infty} \sum_{(\cdot, M_g)^c \cup \{i\}, \mathbf{w}} \sum_{(\cdot, \mathbf{x}^{M_g}) \leq \mathbf{w}} I[\chi_g(M_i, \mathbf{w}) \geq b, \chi_g((M-1)_i, \mathbf{w}) < b] \\
& \prod_{h \in M_g - \{i\}} [r_h^{w_h}(z)p_h^{w_h, x_h}(z, z+v)]f_i^M(z) \\
& (\bar{F}_i^M(z+v)/\bar{F}_i^M(z)) \sum_{(\cdot, M_g, \mathbf{x})} \prod_{h \in M_g^c} r_h^{x_h}(z+v) \sum_{l \neq i} I(x_l > 0)\lambda_l^{x_l, x_l-1}(z+v) \\
& [I(\phi((M-1)_i, (x_l)_l, \mathbf{x}) \geq j)I(\phi(M_i, (x_l-1)_l, \mathbf{x}) \geq j) \\
& - I(\phi((M-1)_i, (x_l-1)_l, \mathbf{x}) \geq j)]dvdz \} \\
& - \sum_{i \in M_g} \sum_{k \in \{1, \dots, M\}} \int_{t=0}^{\infty} \sum_{(\cdot, \mathbf{x})} I[\chi_g(k_i, \mathbf{x}^{M_g}) \geq b, \chi_g((k-1)_i, \mathbf{x}^{M_g}) < b] \\
& [I(\phi(k_i, \mathbf{x}) \geq j) - I(\phi((k-1)_i, \mathbf{x}) \geq j)] \prod_{l \neq i} r_l^{x_l}(t)r_i^k(t)\lambda_i^{k, k-1}(t)dt
\end{aligned}$$

Since $Z_{M_g, b, j} = Z_{I, K, j}$, it follows that $Z_{M_g, b, j} > 0$ implies the existence of $i \in M_g$ and $k \in \{1, \dots, M\}$ such that $Z_{i, k, j} > 0$. The reverse implication, on the other hand, is not true. To see an example of this, let $M_g = \{i, m\}$ be a parallel system of two components. In this special case the module leaves the states $\{b, \dots, M\}$ at $\max(T_{i, b-1}, T_{m, b-1})$. Suppose that $T_{i, b-1} < T_{m, b-1}$, and that the system subsequently leaves the states $\{j, \dots, M\}$ at $T_{l, d-1} > T_{m, b-1}$ for some $l \in M_g^c, d \in \{1, \dots, M\}$. It may then happen that $T'_{i, b-1} > T_{l, d-1}$, i.e. the effect of the minimal repair of component i at $T_{i, b-1}$ extends beyond $T_{l, d-1}$, which again may lead to $Z_{i, b, j} > 0$. At the same time, we may have $T'_{m, b-1} < T_{l, d-1}$, in which case $Z_{M_g, b, j} = Z_{m, b, j} = 0$.

4. Concluding remarks

To work out the lemmas and theorems of this paper has been challenging even only considering nonrepairable systems. From the expressions given in Natvig (2011a) for the Natvig measure for repairable systems in a time interval $[0, t]$, based on expectations, it seems to be over the top to arrive at the corresponding distributions in an analytical form. The answer to this is advanced discrete event simulation methods as applied to multistate network flow systems of repairable components in Huseby and Natvig (2012). As a step one should first work out a simulation program in the case of nonrepairable systems. To be able to arrive at such a program the developments and results of the present paper should be very helpful for instance as a start checking that it produces the correct results for the simple multistate series and parallel systems.

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References

- Barlow RE, Proschan F (1975) Importance of system components and fault tree events. *Stochastic Process Appl* 3:153–173
- Barlow RE, Proschan F (1981) *Statistical theory of reliability and life testing. Probability Models, To Begin With*: Silver Springs, MD
- Birnbaum ZW (1969) On the importance of different components in a multicomponent system. In: Krishnaia PR (ed) *Multivariate analysis-II academic*. New York, pp 581-592.
- Huseby AB, Natvig B (2012) Discrete event simulation methods applied to advanced importance measures of repairable components in multistate network flow systems. Submitted.
- Natvig B (1979) A suggestion of a new measure of importance of system components. *Stochastic Process Appl* 9:319–330
- Natvig B (1982a) On the reduction in remaining system lifetime due to the failure of a specific component. *J Appl Prob* 19:642–652, Correction. *J Appl Prob* 20:713
- Natvig B (1982b) Two suggestions of how to define a multistate coherent system. *Adv Appl Prob* 14:434–455
- Natvig B (1985) New light on measures of importance of system components. *Scand J Statist* 12:43–54
- Natvig B (1990) On information-based minimal repair and the reduction in remaining system lifetime due to the failure of a specific module. *J Appl Prob* 27:365-375
- Natvig B, (2011a) Measures of component importance in nonrepairable and repairable multistate strongly coherent systems. *Methodol Comput Appl Probab* 13:523–547
- Natvig B, (2011b) *Multistate systems reliability theory with applications*. John Wiley & Sons, New York
- Natvig B, Gåsemyr J (2009) New results on the Barlow-Proschan and Natvig measures of component importance in nonrepairable and repairable systems. *Methodol Comput Appl Probab* 11:603–620